Signed Determinantal Point Processes

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Selection models

- Items labeled $1, 2, \ldots, N$.
- Selection model: Random subset of items.
- E.g., ferromagnetic / antiferromagnetic Ising models (statistical physics, e.g., particles with positive spin)
- Objective: Develop a simple model ($\ll 2^N$ parameters), that is tractable, computationally simple and accurate for given applications.
- ML application: Recommender systems.

DPP's

- Model for **random binary vectors** $X = (X_1, \ldots, X_N) \in \{0,1\}^N$
- Equivalently, **random subset** $Y \subseteq [N]$ s.t.

$$\mathbb{P}[J \subseteq Y] = \det(K_J), \forall J \subseteq [N]$$

for some matrix $K \in \mathbb{R}^{N \times N}$.

$K = \begin{pmatrix} & \overset{J}{\leftrightarrow} & \\ & [K_J] & \\ & & \end{pmatrix} \in \mathbb{R}^{N \times N}$

- Ex: $\mathbb{P}[1 \in Y] = K_{1,1}$, $\mathbb{P}[1,2 \in Y] = K_{1,1}K_{2,2} - K_{1,2}K_{2,1}$.
- If $I_N - K$ is invertible, DPP(K) is also an **L-ensemble**:

$$\mathbb{P}[Y = J] = \frac{\det(L_J)}{\det(I_N + L)}, \quad \forall J \subseteq [N]$$

where $L = K(I_N - K)^{-1}$ ($\Leftrightarrow K = L(I_N + L)^{-1}$).

- If K is **symmetric**: DPPs can model **repulsive** interactions: $(X_1, X_2, \ldots, X_N)$ are **negatively associated** ($\gg$ negative correlation), i.e.,

$$cov\big(f(X_i, i \in S), g(X_j, j \in T)\big) \le 0,$$

for all disjoint $S, T \subseteq [N]$ and coordinatewise nondecreasing functions f, g.

E.g., $cov(X_i, X_j) = -K_{i,j}^2 \le 0$.

- In general, $cov(X_i, X_j) = -K_{i,j}K_{j,i}$.
- **Signed DPP**: $K_{i,j} = \varepsilon_{i,j}K_{j,i}$, $\varepsilon_{i,j} \in \{-1,1\}$: $cov(X_i, X_j) = -\varepsilon_{i,j}K_{i,j}^2$

  $\Longrightarrow$ Allow for both **positive** and **negative** dependence.
- **Admissible kernels:** When $I_N - K$ is invertible, DPP(K) is well defined iff $L = K(I_N - K)^{-1}$ is a **P_0-matrix** (i.e., all its principal minors are nonnegative).
- Examples of admissible kernels:
  - Any symmetric K with $0 \le K \le I_N$
  - Any $K = D + \lambda A$ for some $\lambda \in \left(0, \frac{1}{2}\right)$, diagonal matrix D with $D_{i,i} \in [\lambda, 1-\lambda]$ and $A \in [-1,1]^N$.

Identification and learning

Given $Y \sim$ DPP(K), identify and learn K:
The Principal Minor Assignment Problem

- DPP(K) is completely determined by the principal minors of K.
- Given a class $\mathcal{T} \subseteq \mathbb{R}^{N \times N}$, PMA asks:

  I. *What is the collection of all matrices $H \in \mathcal{T}$ that have the same list of principal minors as K ?*

  II. *Given an available list of prescribed principal minors, how to find a matrix $H \in \mathcal{T}$ whose principal minors are given by that list, using as few queries from that list as possible ?*

  I $\Leftrightarrow$ **Identification of K**

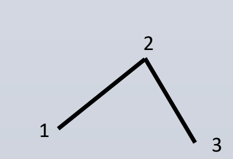
  II $\Leftrightarrow$ **Learning K efficiently**

- Here, $\mathcal{T}$ is the class of **signed kernels**, i.e., $K_{i,j} = \varepsilon_{i,j}K_{j,i}$, $\varepsilon_{i,j} \in \{-1,1\}$.

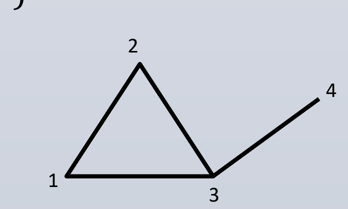
More precisely, given a family $(a_J)_{J \subseteq [N], J \neq \emptyset} \subseteq \mathbb{R}$, characterize $\mathcal{S} = \{H \in \mathcal{T}: det(H_J) = a_J, \forall J \subseteq [N], J \neq \emptyset\}$ and find $H \in \mathcal{S}$ efficiently.

- Main idea:
  - $J = \{i\}$: $H_{i,i} = a_{\{i\}}$.
  - $J = \{i,j\}$: $\varepsilon_{i,j}H_{i,j}^2 = a_{\{i\}}a_{\{j\}} - a_{\{i,j\}}$ $\Longrightarrow$ These determine the **adjacency graph** of any solution H and the values of $\varepsilon_{i,j}$'s.
- **Adjacency graph** of a solution $H \in \mathcal{S}$: $G_H = ([N], E_H)$, with $E_H = \{\{i,j\}: i \neq j, H_{i,j} \neq 0\} = \{\{i,j\}: i \neq j, a_{\{i\}}a_{\{j\}} - a_{\{i,j\}} \neq 0\}$.

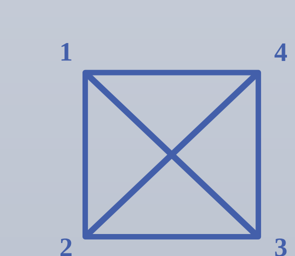
$$H = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$$



$$H = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$



- Remark: $\{i,j\} \notin E_K \Leftrightarrow X_i, X_j$ are independent ($X_i = 1_{i \in Y}$, where $Y \sim$ DPP(K).)
- To recover the signs of $H_{i,j}$'s, use higher order principal minors, associated with small **positive cycles** in G (i.e., $\prod_{\{i,j\} \in C} \varepsilon_{i,j} = 1$)
- Key idea: $\det H_J = \sum_{\sigma \in \mathfrak{S}_J} (-1)^\sigma \prod_{i \in J} H_{i,\sigma(i)}$ $\Longrightarrow$ Decompose each $\sigma \in \mathfrak{S}_J$ as a product of cycles
- Example: If C is an induced cycle (i.e., with no chords) in G_K, with vertex set J, then $\det K_J = F\big(K_{i,i}, K_{i,j}^2 : i,j \in J\big) \pm \big(1 + \prod_{\{i,j\} \in C} \varepsilon_{i,j}\big) \prod_{\{i,j\} \in C} K_{i,j}$
- Ideal situation: There is a basis of induces cycles that are all positive.
- Main issue: Induced cycles may be negative and non induced positive cycles may have more than one positive **traveling**:



$1 \to 2 \to 3 \to 4 \to 1$

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- **Positive travelings** can not be dissociated from the principal minors (e.g., $H_{12}H_{23}H_{34}H_{41} + H_{13}H_{32}H_{24}H_{41} + H_{12}H_{24}H_{43}H_{32}$)
- Assumptions:
  - H is dense, i.e., the graph G_H is complete (this allows to only consider cycles of size 3 and 4)
  - The magnitudes $|H_{i,j}|$ are in *general position* (this allows to separate positive travelings of a given cycle)

$\Longrightarrow$ **Theorem:** The signs can be (not uniquely) recovered using the a_J's, for $\#J \le 4$.

Algorithm

Input: Family $(a_J)_{J \subseteq [N], J \neq \emptyset} \subseteq \mathbb{R}$ of prescribed principal minors.

Output: Matrix $H \in \mathcal{T}$ with $\det H_J = a_J$, $\forall J \subseteq [N], J \neq \emptyset$.

Step 1: Set $\mathrm{H}_{i,i} = a_{\{i\}}$ for all $i \in [N]$.

Step 2: Set $|\mathrm{H}_{i,j}| = |a_{\{i\}}a_{\{j\}} - a_{\{i,j\}}|$ for all $i,j \in [N], i \neq j$.

Step 3: Set $\varepsilon_{i,j} = sign\big(a_{\{i\}}a_{\{j\}} - a_{\{i,j\}}\big)$ for all $i,j \in [N], i \neq j$ s.t. $\mathrm{A}_{i,j} \neq 0$.

Step 4: Find the set $\mathcal{J}^+$ of all triples (i,j,k) such that $\varepsilon_{i,j}\varepsilon_{j,k}\varepsilon_{i,k} = 1$ and find the sign of $H_{i,j}H_{j,k}H_{i,k}$ from $a_{\{i,j,k\}}$.

Step 5: For all $S \subseteq [N]$ if size 4, use a_S in order to find $\prod_{\{i,j\} \in C: i<j} K_{i,j}$ for all the (at most three) positive cycles C that have vertex set S.

Step 6: By Gaussian elimination on $\{+,-\}$, find a sign assignment of all the $H_{i,j}$'s that agree with signs of the products found in Steps 4 and 5.

Conclusions

Theorem: Under the previous assumptions, the set of solutions $\mathcal{S}$ is completely determined by the a_J's, for $\#J \le 4$, and there is a polynomial time algorithm that outputs one solution.

In general, the signs of the $H_{i,j}$'s would be (not uniquely) determined by the a_J's, where J is the vertex set of a positive cycle in some *simple* family of spanning cycles.

Open questions:

1. In general, how to find such a family of spanning cycles efficiently?
2. What properties (analogous to negative association) are satisfied by signed DPP's?
3. For signed DPP's, the eigenstructure of the kernel no longer plays a significant role (e.g., for sampling). How to sample a signed DPP efficiently?

References

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